The Classification of Triangular Semisimple and Cosemisimple Hopf Algebras Over an Algebraically Closed Field

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1 Introduction

In this paper we classify triangular semisimple and cosemisimple Hopf algebras over *any* algebraically closed field k. Namely, we construct, for each positive integer N, relatively prime to the characteristic of k if it is positive, a bijection between the set of isomorphism classes of triangular semisimple and cosemisimple Hopf algebras of dimension N over k, and the set of isomorphism classes of quadruples (G, H, V, u), where G is a group of order N, H is a subgroup of G, V is an irreducible projective representation of H over k of dimension $|H|^{1/2}$, and $u \in G$ is a central element of order ≤ 2. This classification implies, in particular, that *any* triangular semisimple and cosemisimple Hopf algebra over k can be obtained from a group algebra by a twist (it was previously known only in characteristic 0 [EG1, Theorem 2.1]). We also answer positively the question from [EG2] whether the group underlying a *minimal* triangular semisimple Hopf algebra is solvable. We conclude by showing that any triangular semisimple and cosemisimple Hopf algebra over k of dimension bigger than 1 contains a non-trivial grouplike element.

The classification uses Deligne's theorem on Tannakian categories [De] and the results of the paper [M] in an essential way. The proof of solvability and existence of grouplike elements relies on a theorem from [HI], which is proved using the classification of finite simple groups. The classification in positive characteristic relies also on the lifting functor from [EG4].

Throughout the paper, the ground field k is assumed to be algebraically closed.

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2 Twists

Let A be a Hopf algebra over a field k. Recall [Dr1] that a *twist* for A is an invertible element $J \in A \otimes A$ which satisfies

$$(\Delta \otimes I)(J)J_{12} = (I \otimes \Delta)(J)J_{23} \ \ and \ \ (\varepsilon \otimes I)(J) = (I \otimes \varepsilon)(J) = 1, \tag{1}$$

where I is the identity map of A.

If J is a twist for A and x is an invertible element of A then $J^x = \Delta(x)J(x^{-1} \otimes x^{-1})$ is also a twist for A. We will call the twists J and J^x *gauge equivalent*. The element x will be called a *gauge transformation*.

Given a twist J for A, one can define a Hopf algebra $(A^J, \Delta^J, \varepsilon)$ as follows: $A^J = A$ as an algebra, the coproduct is determined by

$$\Delta^J(x) = J^{-1}\Delta(x)J$$

for all $x \in A$, and ε is the ordinary counit of A. If A is triangular with the universal $R-$matrix R, then so is A^J, with the universal $R-$matrix $R^J = J_{21}^{-1}RJ$. It is obvious that two gauge equivalent twists, when applied to a fixed (triangular) Hopf algebra, produce two isomorphic (triangular) Hopf algebras.

Let $A = k[H]$ be the group algebra of a finite group H. We will say that a twist J for A is *minimal* if the right (and left) components of the $R-$matrix $R^J = J_{21}^{-1}J$ span A, i.e. if the corresponding triangular Hopf algebra $(A^J, J_{21}^{-1}J)$ is minimal [R].

Let (A, R) be any triangular semisimple and cosemisimple Hopf algebra over k. Then the Drinfeld element u of A is a grouplike element of order ≤ 2. Moreover, by [LR] in characteristic 0, and by [EG4, Theorem 3.1] in positive characteristic, the square of the antipode of A is the identity map, and hence u is central. Set

$$R_u = \frac{1}{2}(1 \otimes 1 + 1 \otimes u + u \otimes 1 - u \otimes u).$$

Then (A, RR_u) is triangular semisimple and cosemisimple with Drinfeld element 1. This observation allows to reduce questions about triangular semisimple and cosemisimple Hopf algebras over k to the case when the Drinfeld element is 1.

3 Twists for Group Algebras

In this section we will prove the following theorem, which will be used later for classification.

Theorem 3.1 *Let G, G' be finite groups, H, H' subgroups of G, G' respectively, and J, J' minimal twists for $k[H]$, $k[H']$ respectively. Suppose that the triangular Hopf algebras $k[G]^J$, $k[G']^{J'}$ are isomorphic. Then there exists a group isomorphism $\phi : G \to G'$ such that $\phi(H) = H'$, and $(\phi \otimes \phi)(J)$ is gauge equivalent to J' as twists for $k[H']$.*

The rest of the section is devoted to the proof of the theorem.

Lemma 3.2 *Let $\mathcal{C}$ be the category of $k-$representations of a finite group, and F_1, $F_2 : \mathcal{C} \to Vect(k)$ be two fiber functors (i.e. exact and fully faithful symmetric tensor functors, see [DM]) from $\mathcal{C}$ to the category of $k-$vector spaces. Then F_1 is isomorphic to F_2.*

Proof: This is a special case of [DM, Theorem 3.2]. ∎

The following corollary of this lemma answers positively Movshev's question [M, Remark 1] whether any symmetric twist is trivial.

Corollary 3.3 *Let G be a finite group, and J be a symmetric twist for $k[G]$ (i.e. $J_{21} = J$). Then J is gauge equivalent to $1 \otimes 1$.*

Proof: Let $\mathcal{C}$ be the category of representations of G. We have two symmetric tensor structures on the forgetful functor $F : \mathcal{C} \to Vect(k)$; namely, the trivial one and the one defined by J. By Lemma 3.2, the two fiber functors corresponding to these structures are isomorphic. But by definition, an isomorphism between them is an invertible element $x \in k[G]$ such that $J = \Delta(x)(x^{-1} \otimes x^{-1})$. ∎

Remark 3.4 Here is another proof of Corollary 3.3 (in the case when the characteristic of k is relatively prime to $|G|$) which does not use Lemma 3.2 but uses the results of [M]. Consider the G-coalgebra $B_J = k[G]$ with coproduct $\tilde{\Delta}(x) = (x \otimes x)J$, and the dual algebra B_J^*. According to [M], this algebra is semisimple, G acts transitively on its simple ideals, and B_J^*, along with the action of G, completely determines J up to gauge transformations. Clearly, since J is symmetric, this algebra is commutative. So, it is isomorphic, as a G-algebra, to the algebra of functions on a set X on which G acts simply transitively. Corollary 3.3 now follows from the fact that such a G-set is unique up to an isomorphism (the group G itself with G acting by left multiplication). ∎

Lemma 3.5 *Let G, G' be finite groups, J, J' twists for $k[G]$, $k[G']$ respectively, and suppose that the triangular Hopf algebras $k[G]^J$, $k[G']^{J'}$ are isomorphic. Then there exists a group isomorphism $\phi : G \to G'$ such that $(\phi \otimes \phi)(J)$ is gauge equivalent to J'.*

Proof: Let $f : k[G]^J \to k[G']^{J'}$ be an isomorphism of triangular Hopf algebras. Then f defines an isomorphism of triangular Hopf algebras $k[G] \to k[G']^{J'(f\otimes f)(J)^{-1}}$. This implies that the element $J'(f \otimes f)(J)^{-1}$ is a symmetric twist for $k[G']$. Thus, for some invertible $x \in k[G']$ one has $J'(f \otimes f)(J)^{-1} = \Delta(x)(x^{-1} \otimes x^{-1})$. Let $\phi = Ad(x^{-1}) \circ f : k[G] \to k[G']$. It is obvious that ϕ is a Hopf algebra isomorphism, so it comes from a group isomorphism $\phi : G \to G'$. We have $(\phi \otimes \phi)(J) = \Delta(x)^{-1} J'(x \otimes x)$, as desired. ∎

We can now prove Theorem 3.1. By Lemma 3.5, it is sufficient to assume that $G' = G$, and that J is gauge equivalent to J' as twists for $k[G]$, and it is enough to show that there

exists an element $a \in G$ such that $aHa^{-1} = H'$ and $(a \otimes a)J(a^{-1} \otimes a^{-1})$ is gauge equivalent to J' as twists for $k[H']$.

So let $x \in k[G]$ be the invertible element such that $\Delta(x)J(x^{-1} \otimes x^{-1}) = J'$. In particular, this implies that $(x \otimes x)R(x^{-1} \otimes x^{-1}) = R'$, where R, R' are the R-matrices corresponding to J, J' respectively. By the minimality of J, J', we have $xk[H]x^{-1} = k[H']$. Thus,

$$J_0 = \Delta(x)(x^{-1} \otimes x^{-1}) = J'(x \otimes x)J^{-1}(x^{-1} \otimes x^{-1}) \in k[H']^{\otimes 2}.$$

It is obvious that J_0 is a symmetric twist for $k[H']$, so by Corollary 3.3, it is gauge equivalent to $1 \otimes 1$. Thus, $x = x_0 a$, where $x_0 \in k[H']$, and $a \in G$. It is clear that $aHa^{-1} = H'$, and $\Delta(x_0^{-1})J'(x_0 \otimes x_0) = (a \otimes a)J(a^{-1} \otimes a^{-1})$. This concludes the proof of Theorem 3.1.

4 Construction of Triangular Semisimple and Cosemisimple Hopf Algebras from Group-Theoretical Data

Let H be a finite group such that $|H|$ is not divisible by the characteristic of k. Suppose that V is an irreducible projective representation of H over k satisfying $dimV = |H|^{1/2}$. Following the idea of the proof of Proposition 5 in [M], we will construct a twist $J \in k[H] \otimes k[H]$.

Let $\pi : H \to PGL(V)$ be the projective action of H on V, and let $\tilde{\pi} : H \to SL(V)$ be any lifting of this action ($\tilde{\pi}$ need not be a homomorphism). We have $\tilde{\pi}(x)\tilde{\pi}(y) = c(x, y)\tilde{\pi}(xy)$, where c is a 2-cocycle with coefficients in k^*. By [M, Proposition 11], this cocycle is nondegenerate (see [M]) and hence by [M, Proposition 12] the representation of H on $End_k(V)$ is isomorphic to the regular representation.

Remark 4.1 In the paper [M] it is assumed that the characteristic of k is equal to 0, but all the results generalize in a straightforward way to the case when the characteristic of k is positive and relatively prime to the order of the group.

Consider the simple coalgebra $(End_k V)^*$ with comultiplication Δ. Clearly H acts on this coalgebra, and this representation of H is also isomorphic to the regular representation. In particular we can choose an element $\lambda \in (End_k V)^*$ such that the set $\{a \cdot \lambda | a \in H\}$ forms a basis of $(End_k V)^*$, and $< \lambda, I >= 1$ where I is the unit element of $End_k V$. Now, write $\Delta(\lambda) = \sum_{a,b \in H} \gamma(a, b) a \cdot \lambda \otimes b \cdot \lambda$, and set $J = \sum_{a,b \in H} \gamma(a, b) a \otimes b \in k[H] \otimes k[H]$. We claim that J is a twist for $k[H]$.

Indeed, let $\tilde{\Delta} : k[H] \to k[H]^{\otimes 2}$ be determined by $a \mapsto (a \otimes a)J$, and let $f : k[H] \to (End_k V)^*$ be determined by $a \mapsto a \cdot \lambda$. Clearly, f is an isomorphism of $H-$modules which satisfies $\Delta(f(a)) = (f \otimes f)\tilde{\Delta}(a)$. Therefore $(k[H], \tilde{\Delta}, \varepsilon)$ is a coalgebra where $\varepsilon = f^*(I)$, which is equivalent to saying that J satisfies (1). In order to show that J is a twist it remains to show that it is invertible. This is proved in [M, Proposition 13].

It is straightforward to see that two different choices of λ produce two gauge equivalent twists J for $k[H]$, so the equivalence class of J is canonically associated to (H, V).

Now, for any group $G \supset H$, whose order is prime to the characteristic of k, define a triangular semisimple Hopf algebra $F(G,H,V) = (k[G]^J, J_{21}^{-1}J)$. We wish to show that it is also cosemisimple.

Lemma 4.2 *The Drinfeld element of the triangular semisimple Hopf algebra* $(A, R) = F(G,H,V)$ *equals* 1.

Proof: The Drinfeld element u is a grouplike element of A, and for any finite-dimensional A-module V one has $tr|_V(u) = dim_{Rep(A)}V = dimV$ (since $Rep(A)$ is equivalent to $Rep(G)$, see [EG1, Section 1]). In particular, we can set V to be the regular representation, and find that $tr|_A(u) = dim(A) \neq 0$ in k. But it is clear that if g is a non-trivial grouplike element in any finite-dimensional Hopf algebra A, then $tr|_A(g) = 0$. Thus, $u = 1$. ∎

Corollary 4.3 *The triangular semisimple Hopf algebra* $(A, R) = F(G,H,V)$ *is cosemisimple.*

Proof: Since $u = 1$, one has $S^2 = I$ and hence A is cosemisimple (as $dim(A) \neq 0$). ∎

Thus we have assigned a triangular semisimple and cosemisimple Hopf algebra with Drinfeld element $u = 1$ to any triple (G,H,V) as above.

5 The Classification in Characteristic 0

In this section we assume that k is of characteristic 0.

Theorem 5.1 *The assignment* $F : (G,H,V) \mapsto (A,R)$ *is a bijection between isomorphism classes of triples* (G,H,V) *where* G *is a finite group,* H *is a subgroup of* G, *and* V *is an irreducible projective representation of* H *over* k *satisfying* $dimV = |H|^{1/2}$, *and isomorphism classes of triangular semisimple Hopf algebras over* k *with Drinfeld element* $u = 1$.

Proof: We need to construct an assignment F' in the other direction, and check that both $F' \circ F$ and $F \circ F'$ are the identity assignments.

Let (A, R) be a triangular semisimple Hopf algebra over k whose Drinfeld element u is 1. It follows from Deligne's theorem on Tannakian categories (see [EG1, Theorem 2.1]) that there exist finite groups $H \subseteq G$, and a *minimal* twist $J \in k[H] \otimes k[H]$, such that $(A, R) \cong (k[G]^J, J_{21}^{-1}J)$ as triangular Hopf algebras. As we proved in Section 3, these data are unique up to isomorphism and gauge transformations.

Following Movshev [M], define a coalgebra B_J which is $k[H]$ as a vector space, with coproduct $\tilde{\Delta}(x) = (x \otimes x)J$, $x \in H$, and the usual counit. This coalgebra has a natural H-action by left multiplication. It follows from [M] that the coalgebra B_J is simple (see [EG3] for more explanations). Thus, the dual algebra B_J^* is simple as well, and carries an action of H. So we see that B_J^* is isomorphic to $End_k(V)$ for some vector space V, and we

have a homomorphism $\pi : H \to PGL(V)$. Thus V is a projective representation of H. It is shown in [M, Proposition 8] that this representation is irreducible, and it is obvious that $\dim V = |H|^{1/2}$.

It is clear that the isomorphism class of the representation V does not change if J is replaced by a twist J' which is gauge equivalent to J as twists for $k[H]$. Thus, to any isomorphism class of triangular semisimple Hopf algebras (A, R) over k with Drinfeld element 1, we have assigned an isomorphism class of triples (G, H, V). Let us write this as $(G, H, V) = F'(A, R)$.

Thus, we have constructed the map F'.

The identity $F \circ F' = id$ follows from [M, Proposition 5]. The identity $F' \circ F = id$ follows from Theorem 3.1. ∎

Now let (G, H, V, u) be a quadruple, in which (G, H, V) is as above, and u is a central element of G of order ≤ 2. We extend the map F to quadruples by setting $F(G, H, V, u) = (A, RR_u)$, where $(A, R) = F(G, H, V)$.

Theorem 5.2 *The assignment F is a bijection between isomorphism classes of quadruples (G, H, V, u) where G is a finite group, H is a subgroup of G, V is an irreducible projective representation of H over k satisfying $dimV = |H|^{1/2}$, and $u \in G$ is a central element of order ≤ 2, and isomorphism classes of triangular semisimple Hopf algebras over k.*

Proof: Define F' by $F'(A, RR_u) = (F'(A, R), u)$, where $F'(A, R)$, for (A, R) with Drinfeld element 1, is defined in the proof of Theorem 5.1. It is straightforward to see that both $F' \circ F$ and $F \circ F'$ are the identity assignments. ∎

Theorems 5.1 and 5.2 imply the following classification result for minimal triangular semisimple Hopf algebras over k.

Proposition 5.3 *$F(G, H, V, u)$ is minimal if and only if G is generated by H and u.*

Proof: As we have already pointed out, if $(A, R) = F(G, H, V)$ then the sub Hopf algebra $k[H]^J \subseteq A$ is minimal triangular. Therefore, if $u = 1$ then $F(G, H, V)$ is minimal if and only if $G = H$. This obviously remains true for $F(G, H, V, u)$ if $u \neq 1$ but $u \in H$. If $u \notin H$ then it is clear that the $R-$matrix of $F(G, H, V, u)$ generates $k[H']$, where $H' = H \cup uH$. This proves the proposition. ∎

Remark 5.4 As was pointed out already by Movshev, the theory developed in [M] and extended here is an analogue, for finite groups, of the theory of quantization of skew-symmetric solutions of the classical Yang-Baxter equation, developed by Drinfeld [Dr2]. In particular, the operation F is the analogue of the operation of quantization in [Dr2].

6 The Classification in Positive Characteristic

In this section we assume that k is of positive characteristic p, and prove an analogue of Theorem 5.2 by using this theorem itself and the lifting techniques from [EG4].

Let F be the assignment from Section 4. We now have the following.

Theorem 6.1 *The assignment F is a bijection between isomorphism classes of quadruples (G, H, V, u) where G is a finite group of order prime to p, H is a subgroup of G, V is an irreducible projective representation of H over k satisfying $dimV = |H|^{1/2}$, and $u \in G$ is a central element of order ≤ 2, and isomorphism classes of triangular semisimple and cosemisimple Hopf algebras over k.*

Proof: As in the proof of Theorem 5.2 we need to construct the assignment F'.

We recall some notation from [EG4]. Let $\mathcal{O} = W(k)$ be the ring of Witt vectors of k, and K the field of fractions of $\mathcal{O}$ (it is of characteristic 0). Let $\bar{K}$ be the algebraic closure of K.

Let (A, R) be a triangular semisimple and cosemisimple Hopf algebra over k. Lift it (see [EG4]) to a triangular semisimple Hopf algebra $(\bar{A}, \bar{R})$ over K. We have that $(\bar{A} \otimes_K \bar{K}, \bar{R}) = F(G, H, V, u)$. We can now reduce V "$mod\, p$" to get V_p which is an irreducible projective representation of H over the field k. This can be done since V is defined by a nondegenerate $2-$cocycle c (see [M]) with values in roots of unity of degree $|H|^{1/2}$ (as the only irreducible representation of the simple H-algebra with basis $\{X_h | h \in H\}$, and relations $X_g X_h = c(g, h) X_{gh}$). This cocycle can be reduced $mod\, p$ and remains nondegenerate (since the groups of roots of unity of order $|H|^{1/2}$ in k and K are naturally isomorphic), so it defines an irreducible projective representation V_p. Define $F'(A, R) = (G, H, V_p, u)$. It is shown like in characteristic 0 that $F \circ F'$ and $F' \circ F$ are the identity assignments. ∎

Corollary 6.2 *Any triangular semisimple and cosemisimple Hopf algebra over k is obtained from a group algebra by a twist.*

Remark 6.3 Previously the statement of the corollary was only known in characteristic 0 [EG1, Theorem 2.1], and the best result known to us in positive characteristic was [EG4, Theorem 3.9].

Proposition 6.4 *Proposition 5.3 holds in positive characteristic.*

Proof: As before, if $(A, R) = F(G, H, V)$ then the sub Hopf algebra $k[H]^J \subseteq A$ is minimal triangular. This follows from the facts that it is true in characteristic 0, and that the rank of a triangular structure does not change under lifting. Thus, Proposition 5.3 holds in characteristic p. ∎

Remark 6.5 In view of the results of this section, the results of our previous paper [EG3] generalize, without changes, to cotriangular semisimple and cosemisimple Hopf algebras in positive characteristic.

7 The Solvability of the Group Underlying a Minimal Triangular Semisimple Hopf Algebra

A classical fact about complex representations of finite groups is that the dimension of any irreducible representation of a finite group K does not exceed $|K : Z(K)|^{1/2}$, where $Z(K)$ is the center of K. Groups of central type are those groups for which this inequality is in fact an equality. More precisely, a finite group K is said to be of *central type* if it has an irreducible representation V such that $(dimV)^2 = |K : Z(K)|$ (see e.g. [HI]). We shall need the following theorem (conjectured by Iwahori and Matsumoto in 1964) whose proof uses the classification of finite simple groups.

Theorem 7.1 [HI, Theorem 7.3] *Any group of central type is solvable.*

As corollaries, we have the following results.

Corollary 7.2 *Let A be a minimal triangular semisimple Hopf algebra over k, and let G be the finite group such that the categories of representations $Rep(G)$ and $Rep(A)$ are equivalent. Then G is solvable.*

Proof: We may assume that k has characteristic 0 (otherwise we can lift to characteristic 0), and by Proposition 5.3, that the Drinfeld element of A is 1. By Theorem 5.1, the corresponding group G has an irreducible projective representation V with $dimV = |G|^{1/2}$. Let K be a finite central extension of G with central subgroup Z, such that V lifts to a linear representation of K. We have $(\mathrm{dimV})^2 = |K : Z|$. Since $(\mathrm{dimV})^2 \leq |K : Z(K)|$ we get that $Z = Z(K)$ and hence that K is a group of central type. But by Theorem 7.1, K is solvable and hence $G \cong K/Z(K)$ is solvable as well. ∎

Corollary 7.3 *Let A be a triangular semisimple and cosemisimple Hopf algebra over k of dimension bigger than 1. Then A has a non-trivial grouplike element.*

Proof: We can assume that the Drinfeld element u is equal to 1 and that A is not cocommutative. Let A_0 be the minimal part of A. By Corollary 7.2, $A_0 = k[H]^J$ for a solvable group H, $|H| > 1$. Therefore, A_0 has non-trivial 1-dimensional representations. Since $A_0 \cong A_0^{*op}$ as Hopf algebras, we get that A_0, and hence A, has non-trivial grouplike elements. ∎

Corollary 7.3 motivates the following question.

Question 7.4 *Let (A, R) be a quasitriangular semisimple and cosemisimple Hopf algebra over k (e.g. the quantum double of a semisimple and cosemisimple Hopf algebra), and let $dim(A) > 1$. Is it true that A possesses a non-trivial grouplike element?*

Note that a positive answer to this question would imply that for a semisimple and cosemisimple Hopf algebra A over k either A or A^* possesses a non-trivial grouplike element. Such a result is very desirable for the problem of the classification of semisimple Hopf algebras.

8 Group-Theoretical Data Corresponding to Minimal Triangular Hopf Algebras Constructed from a Bijective 1-Cocycle

In this section we determine the group-theoretical data corresponding, under the bijection of the classification given in Theorem 5.1, to the minimal triangular semisimple Hopf algebras constructed in [EG2, Section 4]. We will use the definitions and notation from [EG2].

Let $k = \mathbf{C}$. Let G be a finite group, A be a finite abelian group with a left G-action $(g, a) \mapsto g \cdot a$, and $\pi : G \to A$ be a bijective 1-cocycle, i.e. a bijective map such that $\pi(gg') = \pi(g) + g\pi(g')$ (in particular, $|G| = |A|$). Let $H = G \propto A^*$ be the semidirect product of G by the dual group A^* to A. Following [EG2, Section 4], we can associate to this data the element

$$J = |A|^{-1} \sum_{g \in G, b \in A^*} e^{(\pi(g), b)} b \otimes g$$

(for convenience we use the opposite element to the one from [EG2]). We proved in [EG2] that this element is a minimal twist for $k[H]$, so $k[H]^J$ is a minimal triangular semisimple Hopf algebra with Drinfeld element $u = 1$. Now we wish to find the irreducible projective representation V of H which corresponds to $k[H]^J$ under the correspondence of Theorem 5.1.

We will now construct the representation V and show that it is the one corresponding to $k[H]^J$.

Let $V = Fun(A, k)$ be the space of k-valued functions on A. It has a basis $\{\delta_a | a \in A\}$ of characteristic functions of points. Define a projective action ϕ of H on V by

$$\phi(b)\delta_a = e^{-(a,b)}\delta_a, \ \phi(g)\delta_a = \delta_{g \cdot a + \pi(g)} \text{ and } \phi(bg) = \phi(b)\phi(g)$$

for $g \in G$ and $b \in A^*$. It is straightforward to verify that this is indeed a projective representation.

Proposition 8.1 *The representation V is irreducible, and corresponds to $k[H]^J$ under the bijection of the classification given in Theorem 5.1.*

Proof: Let B_J be the coalgebra $(k[H], \tilde{\Delta})$ where $\tilde{\Delta}(x) = (x \otimes x)J$, $x \in H$. Let B_J^* be the dual algebra. It is enough to show that the H-algebras B_J^* and $End_k(V)$ are isomorphic.

Let us compute the multiplication in the algebra B_J^*. We have

$$\tilde{\Delta}(bg) = |A|^{-1} \sum_{g' \in G, b' \in B} e^{(\pi(g'), b')} b(g \cdot b')g \otimes bgg'. \tag{2}$$

Let $\{Y_{bg}\}$ be the dual basis of B_J^* to the basis $\{bg\}$ of B_J. Let $*$ denote the multiplication law dual to the coproduct $\tilde{\Delta}$. Then, dualizing equation (2), we have

$$Y_{b_2 g_2} * Y_{b_1 g_1} = e^{(\pi(g_1) - \pi(g_2), b_2 - b_1)} Y_{b_1 g_2} \tag{3}$$

for $g_1, g_2 \in G$ and $b_1, b_2 \in A^*$ (here for convenience we write the operations in A and A^* additively). Define $Z_{bg} := e^{(\pi(g),b)}Y_{bg}$ for $g \in G$ and $b \in A^*$. In the basis $\{Z_{bg}\}$ the multiplication law in B_J^* is given by

$$Z_{b_2g_2} * Z_{b_1g_1} = e^{(\pi(g_1),b_2)}Z_{b_1g_2}. \tag{4}$$

Now let us introduce a left action of B_J^* on V. Set

$$Z_{bg}\delta_a = e^{(a,b)}\delta_{\pi(g)}. \tag{5}$$

It is straightforward to check using (4) that (5) is indeed a left action. It is also straightforward to compute that this action is $H-$equivariant. Thus, (5) defines an isomorphism $B_J^* \to End_k(V)$ as $H-$algebras, which proves the proposition. ■

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